

FROM TRIANGULATED CATEGORIES TO MODULE CATEGORIES VIA LOCALISATION II: CALCULUS OF FRACTIONS

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ABSTRACT. We show that the quotient of a Hom-finite triangulated category \mathcal{C} by the kernel of the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$, where T is a rigid object, is preabelian. We further show that the class of regular morphisms in the quotient admit a calculus of left and right fractions. It follows that the Gabriel-Zisman localisation of the quotient at the class of regular morphisms is abelian. We show that it is equivalent to the category of finite dimensional modules over the opposite of the endomorphism algebra of T in \mathcal{C} .

INTRODUCTION

Let k be a field and \mathcal{C} a skeletally small, triangulated Hom-finite k -category which is Krull-Schmidt and has Serre duality. A standard example of such a category is the bounded derived category of finite dimensional modules over a finite dimensional algebra of finite global dimension (see [6]). In this case, the triangulated category is obtained from the abelian category of modules by Gabriel-Zisman (or Verdier) localisation of the quasi-isomorphisms in the bounded homotopy category of complexes of modules.

Here, our approach is the other way around. Given a triangulated category \mathcal{C} as above, we are interested in gaining information about related abelian categories. We are particularly interested in the module categories over (the opposites of) endomorphism algebras of objects in \mathcal{C} . An object T in \mathcal{C} satisfying $\mathrm{Ext}^1(T, T) = 0$ is known as a *rigid* object. In this case it is known [2] that the category of finite dimensional modules over $\mathrm{End}(T)^{op}$ can be obtained as a Gabriel-Zisman localisation of \mathcal{C} , formally inverting the class \mathcal{S} of maps which are inverted by the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$. However, the class \mathcal{S} does not admit a calculus of left or right fractions in the sense of [4, Sect. I.2] (see also [12, Sect. 3]).

If T is a cluster-tilting object then, by a result of Koenig-Zhu [11, Cor. 4.4], the additive quotient $\mathcal{C}/\Sigma T$, where Σ denotes the suspension functor of \mathcal{C} , is equivalent to $\mathrm{mod}\,\mathrm{End}_{\mathcal{C}}(T)^{op}$ (see also [8, Prop. 6.2] and [10, Sect. 5.1]; the case where \mathcal{C} is 2-Calabi-Yau was proved in [10, Prop. 2.1], generalising [3, Thm. 2.2]). However,

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when T is rigid, this is no longer the case in general. It is natural to consider instead the quotient $\mathcal{C}/\mathcal{X}_T$ where \mathcal{X}_T is the class of objects in \mathcal{C} sent to zero by the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$, since, in the cluster-tilting case, $\mathcal{X}_T = \mathrm{add} \Sigma T$. However, one does not obtain the module category this way, since in general $\mathcal{C}/\mathcal{X}_T$ is not abelian.

Our approach here is to show first that $\mathcal{C}/\mathcal{X}_T$ is preabelian, using some arguments generalising those of Koenig-Zhu [11]. This means that, in addition to $\mathcal{C}/\mathcal{X}_T$ being an additive category, every morphism in $\mathcal{C}/\mathcal{X}_T$ has a kernel and a cokernel. This category in general possesses regular morphisms which are not isomorphisms (i.e. morphisms which are both monomorphisms and epimorphisms but which do not have inverses), so it cannot be abelian in general.

However, we show that $\mathcal{C}/\mathcal{X}_T$ does have a nice property. It is *integral*, i.e. the pull-back of any epimorphism (respectively, monomorphism), is again an epimorphism (respectively, monomorphism). This allows us to apply a result of Rump [17, p173] which implies that $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$, the localisation of the category $\mathcal{C}/\mathcal{X}_T$ at the class \mathcal{R} of regular morphisms, is abelian. We assume that \mathcal{C} is skeletally small to ensure that the localisation exists. Furthermore, by the same reference, the class \mathcal{R} admits a calculus of left and right fractions.

We go on to show that the projective objects in $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ are, up to isomorphism, exactly the objects induced by the objects in the additive subcategory of \mathcal{C} generated by T . This implies our main result:

Theorem. *Let \mathcal{C} be a skeletally small, Hom-finite, Krull-Schmidt triangulated category with Serre duality, containing a rigid object T . Let \mathcal{X}_T denote the class of objects X in \mathcal{C} such that $\mathrm{Hom}_{\mathcal{C}}(T, X) = 0$. Let \mathcal{R} denote the class of regular morphisms in $\mathcal{C}/\mathcal{X}_T$. Then \mathcal{R} admits a calculus of left and right fractions. Let $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ denote the localisation of $\mathcal{C}/\mathcal{X}_T$ at \mathcal{R} . Then*

$$(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}} \simeq \mathrm{mod} \mathrm{End}_{\mathcal{C}}(T)^{op}.$$

Let \mathcal{S} denote the class of maps in \mathcal{C} which are inverted by $\mathrm{Hom}_{\mathcal{C}}(T, -)$, and let $\underline{\mathcal{S}}$ denote the image of this class in $\mathcal{C}/\mathcal{X}_T$. Then $\mathcal{R} = \underline{\mathcal{S}}$, and the localisation functor $L_{\mathcal{S}}: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{S}}$ factors through $\mathcal{C}/\mathcal{X}_T$. The main result of [2] was the construction of an equivalence G from $\mathcal{C}_{\mathcal{S}}$ to $\mathrm{mod} \mathrm{End}_{\mathcal{C}}(T)^{op}$, such that $\mathrm{Hom}_{\mathcal{C}}(T, -) = GL_{\mathcal{S}}$. Our theorem above can be seen a refinement of this. It was noted in [2] that \mathcal{S} does not admit a calculus of left or right fractions, thus we can observe that the advantage of passing first to the quotient $\mathcal{C}/\mathcal{X}_T$ is that the subsequent localisation does then admit such a calculus.

We note that [14] contains results obtaining abelian categories as subquotients of triangulated categories; we give an explanation of the relationship between the results obtained here and those in [14] in Section 6. We also remark that A. Beligiannis has recently informed us that, in subsequent work using a different approach, he has been able to generalise our main result to the case of a functorially finite rigid subcategory.

There are interesting parallels between our approach here and the construction of the derived category of an abelian category \mathcal{A} . We follow [9], [5, III.2, III.4].

The derived category of \mathcal{A} can be defined (following Grothendieck) as the Gabriel-Zisman localisation of the category $C(\mathcal{A})$ of complexes over \mathcal{A} at the class of quasi-isomorphisms. This class does not in general admit a calculus of left and right fractions. However, the more commonly used construction (due to Verdier) of the derived category involves passing first to the homotopy category $K(\mathcal{A})$. Then $C(\mathcal{A})$ is a Frobenius category and $K(\mathcal{A})$ is the corresponding stable category, hence a quotient of $C(\mathcal{A})$. Then the class of quasi-isomorphisms in $K(\mathcal{A})$ admits a calculus of left and right fractions. Localising at this class gives rise to the derived category of \mathcal{A} .

In Section 1 we set-up the context in which we work. In Section 2, we recall the definitions of semi-abelian and integral categories and some results of Rump [16, 17] which will be useful. In Section 3, we prove that $\mathcal{C}/\mathcal{X}_T$ is integral. In Section 4, we recall the Gabriel-Zisman theory of localisation and calculi of fractions and also how it can be applied (following Rump [17, Sect. 1]) to the case of the regular morphisms in an integral category. In Section 5, we apply this to $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ to show that it is abelian. By classifying the projective objects in $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$, we deduce the main result. In Section 6, we explain the relationship of the results here to work of Nakaoka [14]. In Section 7, we explain the relationship between our main result and the results in [2].

1. NOTATION

We first set up the context in which we work and define some notation. Let k be a field and \mathcal{C} be a skeletally small, triangulated, Hom-finite, Krull-Schmidt k -category with suspension functor Σ . We need the skeletally small assumption to ensure that the localisations we need exist. We assume that \mathcal{C} has a Serre duality, i.e. an autoequivalence $\nu: \mathcal{C} \rightarrow \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq D \mathrm{Hom}_{\mathcal{C}}(Y, \nu X)$ (natural in X and Y) for all objects X and Y in \mathcal{C} , where D denotes the duality $\mathrm{Hom}_k(-, k)$. Let T be a rigid object in \mathcal{C} and set $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{op}$.

For a full subcategory \mathcal{X} of \mathcal{C} , let

$$\mathcal{X}^{\perp} = \{C \in \mathcal{C} \mid \mathrm{Ext}^1(X, C) = 0 \text{ for each } X \in \mathcal{X}\},$$

and define ${}^{\perp}\mathcal{X}$ dually. For an object X in \mathcal{C} , let $\mathrm{add} X$ denote its additive closure, and let $X^{\perp} = (\mathrm{add} X)^{\perp}$. A rigid object T is called *cluster-tilting* if $\mathrm{add} T = T^{\perp}$. Let $\mathcal{X}_T = (\Sigma T)^{\perp}$.

We also recall the triangulated version of Wakamatsu's Lemma; see e.g. [8, Section 2].

Lemma 1.1. *Let \mathcal{X} be an extension-closed subcategory of a triangulated category \mathcal{C} .*

- (a) *Suppose that $X \rightarrow C$ is a minimal right \mathcal{X} -approximation of C and $\Sigma^{-1}C \rightarrow Y \rightarrow X \rightarrow C$ a completion to a triangle. Then Y is in \mathcal{X}^{\perp} , and the map $\Sigma^{-1}C \rightarrow Y$ is a left \mathcal{X}^{\perp} -approximation of $\Sigma^{-1}C$.*
- (b) *Suppose that $C \rightarrow X$ is a minimal left \mathcal{X} -approximation of C and $\Sigma^{-1}Z \rightarrow C \rightarrow X \rightarrow Z \rightarrow \Sigma C$ a completion to a triangle. Then Z is in ${}^{\perp}\mathcal{X}$, and the map $Z \rightarrow \Sigma C$ is a right ${}^{\perp}\mathcal{X}$ -approximation of ΣC .*

Using this, we obtain:

Lemma 1.2. *Let T be a rigid object in \mathcal{C} . Then the subcategory \mathcal{X}_T of \mathcal{C} is functorially finite.*

Proof. This follows from combining Wakamatsu's Lemma (Lemma 1.1) with the existence of Serre duality. \square

2. PREABELIAN CATEGORIES

Recall that an additive category \mathcal{A} is said to be *preabelian* if every morphism has a kernel and a cokernel. In this section we shall recall some of the theory of preabelian categories that we need in order to study $\mathcal{C}/\mathcal{X}_T$. A morphism is said to be *regular* (or a bimorphism) if it is both an epimorphism and a monomorphism.

According to [17, Sect. 1] a preabelian category is called *left semi-abelian* (respectively, *right semi-abelian*) if every morphism f has a factorisation of the form ip where p is a cokernel and i is a monomorphism (respectively, where p is an epimorphism and i is a kernel); see [17, Sect. 1], where it is pointed out that in the left semi-abelian case p is necessarily $\text{coim}(f) = \text{coker}(\ker(f))$ and in the right semi-abelian case i is necessarily $\text{im}(f) = \ker(\text{coker}(f))$. A preabelian category is said to be *semi-abelian* if it is both left and right semi-abelian.

We remark that pullbacks and pushouts always exist in a preabelian category. For the pullback of maps $c: B \rightarrow D$ and $d: C \rightarrow D$, we can take the kernel of the map $B \amalg C \rightarrow D$ whose components are c and $-d$, obtaining a pullback diagram:

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow b & & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

There is a dual construction for the pushout.

We recall the following characterisation of semi-abelian categories:

Proposition 2.1. [17, Prop. 1] *Let \mathcal{A} be a preabelian category. Then \mathcal{A} is left semi-abelian if and only if, in any pullback diagram as above, a is an epimorphism whenever d is a cokernel.*

A dual characterisation in terms of pushout diagrams holds for right semi-abelian categories.

A preabelian category is said to be *left integral* provided that, in any pullback diagram as above, a is an epimorphism whenever d is an epimorphism. A dual definition involving pushouts is used to define right integral categories. A preabelian category which is both left and right integral is said to be *integral*.

The following then follows from Proposition 2.1.

Proposition 2.2. [17, Cor. 1] *Any left integral (respectively, integral) category is left semi-abelian (respectively, semi-abelian).*

We recall the following two results from [17].

Lemma 2.3. [17, Lemma 1] *Let \mathcal{A} be a preabelian category. In a pullback diagram (1), whenever d is a monomorphism, a is a monomorphism.*

We also recall the following (which also includes a dual statement involving pushouts which we will not need here).

Proposition 2.4. [17, Prop. 6] *Let \mathcal{A} be a semi-abelian category. Then the following are equivalent.*

- (a) *The category \mathcal{A} is integral.*
- (b) *For any pullback diagram (1), a is regular whenever d is regular.*

Finally, we note the following, which is easy to show using the definitions.

Lemma 2.5. *Let h be a map in an additive category which is a weak cokernel of a map g and an epimorphism. Then h is a cokernel of g .*

3. PROPERTIES OF $\mathcal{C}/\mathcal{X}_T$

In this section, we consider the factor category $\mathcal{C}/\mathcal{X}_T$. The objects in $\mathcal{C}/\mathcal{X}_T$ are the same as those in \mathcal{C} . For objects X, Y in \mathcal{C} , $\text{Hom}_{\mathcal{C}/\mathcal{X}_T}(X, Y)$ is given by $\text{Hom}_{\mathcal{C}}(X, Y)$ modulo morphisms factoring through \mathcal{X}_T . We denote the image of a morphism f in \mathcal{C} by \underline{f} . Note that, since \mathcal{C} is k -additive, so is $\mathcal{C}/\mathcal{X}_T$.

We will show the following result, which can be regarded as a generalisation of [11, Theorem 3.3]. Our proof is inspired by the proof in [11].

Theorem 3.1. *The factor category $\mathcal{C}/\mathcal{X}_T$ is preabelian.*

In order to prove Theorem 3.1, we will need the following lemmas.

Lemma 3.2. *Consider a commutative diagram*

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma A \\ \downarrow \delta_1 & & \downarrow \delta_2 & & \downarrow \delta_3 & & \downarrow \Sigma \delta_1 \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & \Sigma A' \end{array}$$

in a triangulated category, where the rows are triangles.

- (a) *If the composition $\delta_2\alpha$ vanishes, then there are maps $\epsilon_1: C \rightarrow B'$ and $\epsilon_2: \Sigma A \rightarrow C'$, such that $\delta_3 = \beta'\epsilon_1 + \epsilon_2\gamma$.*
- (b) *If the composition $\gamma'\delta_3$ vanishes, then there are maps $\phi_1: B \rightarrow A'$ and $\phi_2: C \rightarrow B'$ such that $\delta_2 = \phi_2\beta + \alpha'\phi_1$.*

For a map $f: X \rightarrow Y$ in \mathcal{C} , consider the triangle $\Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$.

Lemma 3.3. (a) *The map $\underline{f}: X \rightarrow Y$ is a monomorphism if and only if $\underline{h} = 0$.*
 (b) *The map $\underline{f}: X \rightarrow Y$ is an epimorphism if and only if $\underline{g} = 0$.*
 (c) *The map $\underline{f}: X \rightarrow Y$ is regular if and only if $\underline{g} = 0 = \underline{h}$.*

Proof. (c) follows by definition from (a) and (b). We prove only (b), the proof of (a) being dual.

We consider first the case when Z is in \mathcal{X}_T . We then need to show that \underline{f} is an epimorphism in $\mathcal{C}/\mathcal{X}_T$.

Let $p: Y \rightarrow M$ be a map such that $\underline{p}f = 0$. Then there is an object U' in \mathcal{X}_T , and a commuting square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow p \\ U' & \longrightarrow & M \end{array}$$

By Lemma 1.2, a minimal right \mathcal{X}_T -approximation $U \rightarrow M$ exists. The map $U' \rightarrow M$ factors through $U \rightarrow M$, hence there is also a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & & \downarrow p \\ U & \xrightarrow{f'} & M \end{array}$$

which we extend to a commutative diagram

$$\begin{array}{ccccccc} \Sigma^{-1}Z & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & Z \\ \downarrow & & \downarrow q & & \downarrow p & & \downarrow \\ \Sigma^{-1}N & \longrightarrow & U & \xrightarrow{f'} & M & \longrightarrow & N \end{array}$$

where the rows are triangles. By Wakamatsu's Lemma (Lemma 1.1), we have that $\Sigma^{-1}N$ is in \mathcal{X}_T^\perp . Therefore the map $Z \rightarrow N$ is zero, using that Z is by assumption in \mathcal{X}_T . By commutativity, the composition $\Sigma^{-1}Z \rightarrow X \rightarrow U$ vanishes. Hence, we have by Lemma 3.2, that there are maps $v_1: Y \rightarrow U$, and $v_2: Z \rightarrow M$, such that $p = f'v_1 + v_2g$. Hence p factors through $U \amalg Z$, which is in \mathcal{X}_T , so we have $\underline{p} = 0$.

Now consider the general case, so assume g factors through an object \bar{V} in \mathcal{X}_T and consider the induced commutative diagram

$$\begin{array}{ccccccc} \Sigma^{-1}V & \longrightarrow & N & \xrightarrow{f'} & Y & \longrightarrow & V \\ \downarrow & & \downarrow r & & \parallel & & \downarrow \\ \Sigma^{-1}Z & \xrightarrow{h} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

where the rows are triangles. Now $f' = fr$ and hence $\underline{f'} = \underline{f}r$. Note that since V is in \mathcal{X}_T , we have that $\underline{f'}$ is an epimorphism. It follows that $\underline{f}r$, and hence \underline{f} , is an epimorphism.

Conversely, if \underline{f} is an epimorphism then, since $gf = 0$ we have $\underline{g}\underline{f} = 0$, so $\underline{g} = 0$. \square

Lemma 3.4. *For any map $f: X \rightarrow Y$ in \mathcal{C} , the map $\underline{f}: X \rightarrow Y$ has a kernel and a cokernel.*

Proof. We construct a cokernel of \underline{f} . The construction of a kernel is dual.

Consider a minimal right add T -approximation $a: T_0 \rightarrow X$. Compose this with f , and complete the composition fa to a triangle

$$(2) \quad T_0 \rightarrow Y \xrightarrow{c} M \rightarrow \Sigma T_0$$

By the octahedral axiom, there is a commutative diagram

$$\begin{array}{ccccccc} T_0 & \longrightarrow & Y & \xrightarrow{c} & M & \longrightarrow & \Sigma T_0 \\ a \downarrow & & \parallel & & \downarrow b & & \downarrow \Sigma a \\ X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

We claim that \underline{c} is a cokernel for \underline{f} .

Consider a map $p: Y \rightarrow N$, such that $\underline{pf} = 0$. Assume pf factors through an object U in \mathcal{X}_T , so there is a commuting square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{r} & N \end{array}$$

Extend this to a commuting diagram of triangles, and compose with the previous map of triangles, to obtain the diagram

$$\begin{array}{ccccccc} T_0 & \longrightarrow & Y & \xrightarrow{c} & M & \longrightarrow & \Sigma T_0 \\ \downarrow & & \parallel & & \downarrow b & & \downarrow \Sigma a \\ X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow p & & \downarrow d & & \downarrow \\ U & \xrightarrow{r} & N & \longrightarrow & Z' & \longrightarrow & \Sigma U \end{array}$$

The composition $T_0 \rightarrow X \rightarrow U$ vanishes, since U is in \mathcal{X}_T . Hence the composition $M \xrightarrow{b} Z \xrightarrow{d} Z' \rightarrow \Sigma U$ also vanishes. Now Lemma 3.2 implies that there exist maps $e_1: Y \rightarrow U$ and $e_2: M \rightarrow N$, such that $p = re_1 + e_2c$. Since U is in \mathcal{X}_T , this implies $\underline{p} = \underline{e_2c}$.

This shows that \underline{c} is a weak cokernel for \underline{f} . It is also clear that \underline{c} is an epimorphism, using the triangle (2). It then follows from Lemma 2.5 that \underline{c} is actually a cokernel for \underline{f} . \square

Proof of Theorem 3.1. The additivity of $\mathcal{C}/\mathcal{X}_T$ follows directly from the additivity of \mathcal{C} . By Lemma 3.4, we have that for any map $f: X \rightarrow Y$, the induced map \underline{f} has both a kernel and a cokernel. \square

In order to show that $\mathcal{C}/\mathcal{X}_T$ is also integral, we need to study its projective objects. According to [13], an object P in a preabelian category (indeed, in any category) is

said to be *projective* if, for any epimorphism $c: B \rightarrow C$, any morphism $f: P \rightarrow C$ factors through c :

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ B & \xrightarrow{c} & C \end{array}$$

We shall use this definition. But we note that Rump [17, p170] uses a different definition; the above diagram should commute only for any cokernel c . Such objects are referred to as *quasi-projectives* in [15, Defn. 7.5.2] and we shall use this terminology. Note that the two notions are the same in an abelian category, as then epimorphisms and cokernels coincide. The dual objects will be referred to as *quasi-injectives*.

In the following three proofs, we use some arguments based on the proof of [11, Theorem 4.3].

Lemma 3.5. *Every object in $\text{add } T$, when regarded as an object in $\mathcal{C}/\mathcal{X}_T$, is projective.*

Proof. Let $\underline{f}: X \rightarrow Y$ be an epimorphism in $\mathcal{C}/\mathcal{X}_T$, and $\underline{u}: T_0 \rightarrow Y$ any morphism, where T_0 lies in $\text{add } T$. Completing f to a triangle in \mathcal{C} , we have the diagram:

$$\begin{array}{ccccc} & T_0 & & & \\ & \downarrow u & & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \longrightarrow \Sigma X \end{array}$$

Since \underline{f} is an epimorphism, by Lemma 3.3 we have that g factors through \mathcal{X}_T . Hence $gu = 0$, so u factors through f and thus \underline{u} factors through \underline{f} as required. \square

Lemma 3.6. *The category $\mathcal{C}/\mathcal{X}_T$ has enough projectives.*

Proof. Let X be an object in $\mathcal{C}/\mathcal{X}_T$. and let $f: T_0 \rightarrow X$ be a minimal right $\text{add } T$ approximation of X in \mathcal{C} . Complete it to a triangle:

$$U \longrightarrow T_0 \xrightarrow{f} X \longrightarrow \Sigma U.$$

By Wakamatsu's Lemma (see Lemma 1.1), U lies in T^\perp , so ΣU lies in $\Sigma T^\perp = \mathcal{X}_T$. Hence, by Lemma 3.3, $\underline{f}: T_0 \rightarrow X$ is an epimorphism, as required. It now follows from Lemma 3.5 that $\mathcal{C}/\mathcal{X}_T$ has enough projectives. \square

Dually, it can be shown that:

Lemma 3.7. (a) *Every object in $\text{add } \Sigma^2 T$, regarded as an object in $\mathcal{C}/\mathcal{X}_T$, is injective.*

(b) *The category $\mathcal{C}/\mathcal{X}_T$ has enough injectives.*

We recall:

Proposition 3.8. [17, Cor. 2]

If \mathcal{A} is a preabelian category with enough quasi-projectives (respectively, quasi-injectives), then \mathcal{A} is left (respectively, right) semi-abelian.

It already follows from Proposition 3.8 that $\mathcal{C}/\mathcal{X}_T$ is semi-abelian, using Lemmas 3.6 and 3.7. However, a modification of the argument in the proof of this result allows us to show:

Proposition 3.9. Let \mathcal{A} be a preabelian category.

- (1) Suppose that \mathcal{A} has enough projectives. Then \mathcal{A} is left integral.
- (2) Suppose that \mathcal{A} has enough injectives. Then \mathcal{A} is right integral.
- (3) Suppose that \mathcal{A} has enough projectives and enough injectives. Then \mathcal{A} is integral.

Proof. As remarked above, we use an approach similar to the proof of [17, Cor. 2]. For (a), suppose we are given a pullback diagram:

$$(3) \quad \begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow \underline{b} & & \downarrow \underline{c} \\ C & \xrightarrow{\underline{d}} & D \end{array}$$

in $\mathcal{C}/\mathcal{X}_T$ with \underline{d} an epimorphism. Since \mathcal{A} has enough projectives, there is an epimorphism $\underline{a}': P \rightarrow B$ in $\mathcal{C}/\mathcal{X}_T$ where P is projective in \mathcal{A} . Since \underline{d} is an epimorphism, there is a map $\underline{b}': P \rightarrow C$ in $\mathcal{C}/\mathcal{X}_T$ such that $\underline{c}\underline{a}' = \underline{d}\underline{b}'$. Since the diagram (3) is a pullback, there is a map $\underline{e}: P \rightarrow A$ such that $\underline{a}\underline{e} = \underline{a}'$ and $\underline{b}\underline{e} = \underline{b}'$. Since \underline{a}' is an epimorphism, so is \underline{a} , and (a) follows.

$$\begin{array}{ccccc} & & P & & \\ & & \searrow \underline{e} & & \searrow \underline{a}' \\ & & A & \xrightarrow{a} & B \\ & & \downarrow \underline{b} & & \downarrow \underline{c} \\ & & C & \xrightarrow{\underline{d}} & D \end{array}$$

The proof of (b) is dual to the proof of (a), and (c) follows from (a) and (b). \square

We also remark that, for a semi-abelian category, left integrality is equivalent to right integrality (see [17, Cor. p173]).

Corollary 3.10. The category $\mathcal{C}/\mathcal{X}_T$ is integral.

Proof. This follows from Lemmas 3.5, 3.6 and 3.7, together with Proposition 3.9. \square

4. LOCALISATION

Let \mathcal{D} be a category. A class \mathcal{R} of morphisms in \mathcal{D} is said to admit a *calculus of right fractions* [4, I.2] provided that the following holds:

- (RF1) The identity morphisms of \mathcal{D} lie in \mathcal{R} and \mathcal{R} is closed under composition.

(RF2) Any diagram of the form:

$$\begin{array}{ccc} & B & \\ & \downarrow f & \\ C & \xrightarrow{r} & D \end{array}$$

with $r \in \mathcal{R}$ has a completion to a commuting square of the following form:

$$\begin{array}{ccc} A & \xrightarrow{r'} & B \\ f' \downarrow & & \downarrow f \\ C & \xrightarrow{r} & D \end{array}$$

with r' in \mathcal{R} .

(RF3) If $r: Y \rightarrow Y'$ lies in \mathcal{R} and $f, f': X \rightarrow Y$ are maps such that $rf = rf'$ then there is a map $r': X' \rightarrow X$ in \mathcal{R} such that $fr' = f'r'$.

There is a dual set of axioms, (LF1-3), for left fractions. Let us assume that \mathcal{D} is skeletally small, so that the Gabriel-Zisman localisation $\mathcal{D}_{\mathcal{R}}$ of \mathcal{D} at \mathcal{R} exists. In this situation, $\mathcal{D}_{\mathcal{R}}$ has a very nice description; see [4, I.2] or [12, Sect. 3]. The objects in $\mathcal{D}_{\mathcal{R}}$ are the same as the objects of \mathcal{D} . The morphisms from X to Y are *right fractions*, of the form

$$X \xleftarrow{r} A \xrightarrow{f} Y$$

denoted $[r, f]_{\text{RF}}$, up to an equivalence relation: two such fractions $[r, f]_{\text{RF}}$ and $[r', f']_{\text{RF}}$ are equivalent if there is a commutative diagram of the form:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow r & \uparrow & \searrow f & \\ X & \xleftarrow{r''} & A'' & \xrightarrow{f''} & Y \\ & \swarrow r' & \downarrow & \searrow f' & \\ & & A' & & \end{array}$$

where r'' lies in \mathcal{R} .

The composition of two right fractions $[r', f']_{\text{RF}} \circ [r, f]_{\text{RF}}$ is given by the right fraction $[rr'', f'f'']_{\text{RF}}$ where rr'' , a morphism in \mathcal{R} , and $f'f''$, a morphism in \mathcal{C} , are obtained from an application of axiom (RF2) which gives rise to the following commutative diagram:

$$\begin{array}{ccccc} & & C & & \\ & \swarrow r'' & & \searrow f'' & \\ & A & & B & \\ r \swarrow & & f \searrow & r' \swarrow & f' \searrow \\ X & & Y & & Z \end{array}$$

The localisation functor from \mathcal{D} to $\mathcal{D}_{\mathcal{R}}$ takes a morphism f to $[id, f]_{\text{RF}}$. We shall denote this image by $[f]$. For $r \in \mathcal{R}$, $[r, id]_{\text{RF}}$ is the inverse of $[r]$ (i.e. the formal inverse adjoined in the localisation). We shall denote it x_r . Thus, every morphism in $\mathcal{D}_{\mathcal{R}}$ has the form $[r, f]_{\text{RF}} = [f]x_r$, where f is a morphism in \mathcal{D} and r lies in \mathcal{R} . Similarly, if \mathcal{D} satisfies (LF1-3), there is a dual description of $\mathcal{D}_{\mathcal{R}}$ by left fractions, and so every morphism in $\mathcal{D}_{\mathcal{R}}$ can be written in the form $[g, s]_{\text{LF}} = x_s[g]$, where g is a morphism in \mathcal{D} and s lies in \mathcal{R} .

According to [17, p173], the following result holds. We include a proof for the convenience of the reader.

Proposition 4.1. [17, p173] *Let \mathcal{A} be a semi-abelian category. Then \mathcal{A} is integral if and only if the class \mathcal{R} of regular morphisms in \mathcal{A} admits a calculus of right fractions and a calculus of left fractions.*

Proof. We firstly note that it follows from the definitions that \mathcal{R} satisfies (RF1). Let r, f, f' be as in (RF3) above and suppose that $rf = rf'$. Then $r(f - f') = 0$. Since r is regular, it is a monomorphism, so $f - f' = 0$ and $f = f'$. Thus we can just take r' to be the identity map on X and we see that (RF3) is satisfied.

We will now show that \mathcal{A} is left integral if and only if (RF2) holds. Suppose first that \mathcal{A} is left integral and we are given a diagram:

$$(4) \quad \begin{array}{ccc} & B & \\ & \downarrow f & \\ C & \xrightarrow{r} & D \end{array}$$

with r regular. Let

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ b \downarrow & & \downarrow f \\ B & \xrightarrow{r} & D \end{array}$$

be the pullback of this diagram. Since r is an epimorphism and \mathcal{A} is left integral, a is also an epimorphism. Since r is a monomorphism, a is a monomorphism by Lemma 2.3. Hence a is also regular and we see that (RF2) holds.

Conversely, suppose that (RF2) holds and consider a pullback diagram of the form:

$$(5) \quad \begin{array}{ccc} A & \xrightarrow{a} & B \\ b \downarrow & & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

with d regular. By (RF2), there is a commuting diagram

$$\begin{array}{ccc} A' & \xrightarrow{a'} & B \\ b' \downarrow & & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

with a' regular. Since the diagram (5) is a pullback, we have a map $e: A' \rightarrow A$ making the diagram:

(6)

$$\begin{array}{ccccc} A' & & \xrightarrow{a'} & & B \\ & \searrow e & & \searrow a & \\ & & A & \xrightarrow{a} & B \\ & \searrow b' & \downarrow b & & \downarrow c \\ & & C & \xrightarrow{d} & D \end{array}$$

commute. Since a' is regular, it is an epimorphism, so a is also an epimorphism. Again by Lemma 2.3, the fact that d is a monomorphism implies that a is also a monomorphism. Hence a is regular. By Proposition 2.4, \mathcal{A} is integral, hence left integral.

Thus we have seen that \mathcal{A} is left integral if and only if (RF2) holds, if and only if \mathcal{R} admits a calculus of right fractions. A similar argument shows that \mathcal{A} is right integral if and only if \mathcal{R} admits a calculus of left fractions. The result is proved. \square

We note that the proof shows that in fact:

Corollary 4.2. *Let \mathcal{A} be a semi-abelian category. Then \mathcal{A} is integral if and only if the class \mathcal{R} of regular morphisms in \mathcal{A} admits a calculus of right fractions (respectively, a calculus of left fractions).*

Proof. If \mathcal{A} is integral then it is left integral. The proof of Proposition 4.1 shows that then RF1-3 are satisfied by \mathcal{R} and conversely that if RF1-3 are satisfied then \mathcal{A} is integral. The statement for right fractions follows, and a dual argument shows the statement for left fractions. \square

In the rest of this section, we assume that \mathcal{A} is skeletally small, so that localisations exist.

Remark 4.3. *We note that, by [4, 3.3, Cor. 2], the localisation of \mathcal{A} at \mathcal{R} in the situation of Proposition 4.1 is additive, since \mathcal{A} is additive and \mathcal{R} admits a calculus of right fractions. Furthermore, the localisation functor is additive. See also [19, 10.3.11].*

Lemma 4.4. *Let \mathcal{A} be an integral category and let \mathcal{R} be the class of regular morphisms in \mathcal{A} . Then the localisation functor $L: \mathcal{A} \rightarrow \mathcal{A}_{\mathcal{R}}$ is faithful.*

Proof. By Corollary 4.2 (and recalling 2.2), \mathcal{R} admits a calculus of right fractions. Let $f: X \rightarrow Y$ be a morphism in \mathcal{A} and suppose that $[f] = 0$. Then we have a commutative diagram:

$$\begin{array}{ccccc}
 & & A & & \\
 & \swarrow id & \uparrow u & \searrow f & \\
 X & \xleftarrow{r''} & A'' & \xrightarrow{f''} & Y \\
 & \swarrow id & \downarrow v & \searrow 0 & \\
 & & A' & &
 \end{array}$$

We see that $r'' = u = v$ is regular, and $fu = 0$. Since u is an epimorphism, $f = 0$ as required. \square

We remark that, as a consequence, $[r, f]_{\text{RF}} = [g, s]_{\text{LF}}$ if and only if $[f]x_r = x_s[g]$, if and only if $[sf] = [gr]$, if and only if $sf = gr$, as noted in [17, p173].

We note that:

Lemma 4.5. *Let \mathcal{A} be an integral category and let \mathcal{R} be the class of regular morphisms in \mathcal{A} . Then a morphism f in \mathcal{A} is an epimorphism if and only if $[f]$ is an epimorphism. It is a monomorphism if and only if $[f]$ is a monomorphism.*

Proof. Suppose first that $[f]$ is an epimorphism and g is a morphism in \mathcal{A} for which $gf = 0$. Then $[g][f] = [gf] = 0$. Hence $[g] = 0$ since $[f]$ is an epimorphism. By Lemma 4.4, $g = 0$, so f is an epimorphism. Conversely, suppose that f is an epimorphism and that $(x_r[g])[f] = 0$ for a morphism g in \mathcal{A} and a regular morphism r in \mathcal{A} . Then $[g][f] = 0$, so $[gf] = 0$, so by Lemma 4.4, $gf = 0$. Hence $g = 0$, so $[g] = 0$ and therefore $x_r[g] = 0$ as required. So $[f]$ is an epimorphism. The monomorphism case is proved similarly. The result is proved. \square

Lemma 4.6. *Let \mathcal{A} be an integral category and let \mathcal{R} be the class of regular morphisms in \mathcal{A} . Let f, r be morphisms in \mathcal{A} , with r regular, and let c be a cokernel of f in \mathcal{A} . Then $[c]$ is a cokernel of $[f]x_r$ in $\mathcal{A}_{\mathcal{R}}$. Similarly, if j is a kernel of f in \mathcal{A} then $[j]$ is a kernel of $x_r[f]$ in $\mathcal{A}_{\mathcal{R}}$.*

Proof. We have $[c][f]x_r = [cf]x_r = [0]x_r = 0$. Suppose that g, s are morphisms in \mathcal{A} , with s regular, and $(x_s[g])([f]x_r) = 0$. Then $x_s[gf]x_r = 0$, so $[gf] = 0$, so $gf = 0$ in \mathcal{A} . Hence g factors through c , so $[g]$ factors through $[c]$, so $x_s[g]$ factors through $[c]$. Hence $[c]$ is a weak cokernel of $[f]x_r$. Since c is an epimorphism in \mathcal{A} , it follows from Lemma 4.5 that $[c]$ is an epimorphism in \mathcal{A} . Therefore, by Lemma 2.5, $[c]$ is a cokernel of $[f]x_r$. The result for kernels is proved similarly. \square

We recall that every morphism $f: X \rightarrow Y$ in a preabelian category \mathcal{A} has a factorisation of the form:

$$X \xrightarrow{u} \text{coim}(f) \xrightarrow{\tilde{f}} \text{im}(f) \xrightarrow{v} Y$$

and \mathcal{A} is abelian if and only if \tilde{f} is an isomorphism for all morphisms f in \mathcal{A} .

Lemma 4.7. [17, p167] *Let \mathcal{A} be a preabelian category. Then \mathcal{A} is semi-abelian if and only if \tilde{f} is regular for all morphisms f in \mathcal{A} .*

Proof. By definition (see Section 2), if \mathcal{A} is semi-abelian then every morphism f has a factorisation of the form ip where $p = \text{coim}(f)$ and i is a monomorphism. Comparing this with the factorisation above we see that $i = v\tilde{f}$ and thus, since i is a monomorphism, so is \tilde{f} . Dually, we see that \tilde{f} is an epimorphism, and hence regular. Conversely, suppose that for all morphisms f in \mathcal{A} , \tilde{f} is regular. Then, in the factorisation above, $v\tilde{f}$ must be a monomorphism as v and \tilde{f} are. Dually, $\tilde{f}u$ is an epimorphism and we see that \mathcal{A} is semi-abelian as required. \square

According to [17], we have the following theorem. Again we give details for the interested reader.

Theorem 4.8. [17, p173] *Let \mathcal{A} be an integral category. Then the localisation $\mathcal{A}_{\mathcal{R}}$ (if it exists) of \mathcal{A} at the class of regular morphisms is an abelian category.*

Proof. As we have already observed, by [4, 3.3, Cor. 2], $\mathcal{A}_{\mathcal{R}}$ is an additive category. By Lemma 4.6, $\mathcal{A}_{\mathcal{R}}$ is preabelian. Since \mathcal{A} is semi-abelian (by Proposition 2.2), it follows from Lemma 4.7 that in the factorisation:

$$X \xrightarrow{u} \text{coim}(f) \xrightarrow{\tilde{f}} \text{im}(f) \xhookrightarrow{v} Y$$

of any morphism f in \mathcal{A} , \tilde{f} is regular. It is easy to check that applying the localisation functor to this factorisation gives the corresponding factorisation of $[f]$. It follows that for morphisms of the form $\alpha = [f]$ in $\mathcal{A}_{\mathcal{R}}$, $\tilde{\alpha}$ is invertible. Since every morphism in $\mathcal{A}_{\mathcal{R}}$ can be obtained by composing a morphism of this form with an invertible morphism in $\mathcal{A}_{\mathcal{R}}$, it follows that \tilde{u} is invertible for all morphisms u in $\mathcal{A}_{\mathcal{R}}$ and hence that $\mathcal{A}_{\mathcal{R}}$ is abelian as required. \square

5. THE LOCALISATION OF $\mathcal{C}/\mathcal{X}_T$ IS EQUIVALENT TO $\text{mod } \Gamma$.

In this section we will show that $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ is isomorphic to $\text{mod } \Gamma$, where as before $\Gamma = \text{End}_{\mathcal{C}}(T)^{op}$.

We have seen (Corollary 3.10) that $\mathcal{C}/\mathcal{X}_T$ is an integral category. Since we assume \mathcal{C} is skeletally small, $\mathcal{C}/\mathcal{X}_T$ is also skeletally small. Applying Theorem 4.8 to the integral category $\mathcal{C}/\mathcal{X}_T$ we see that $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ is abelian:

Theorem 5.1. *Let \mathcal{C} be a skeletally small, Hom-finite, Krull-Schmidt triangulated category with Serre duality, containing a rigid object T . Let \mathcal{X}_T denote the class of objects X in \mathcal{C} such that $\text{Hom}_{\mathcal{C}}(T, X) = 0$. Then the class \mathcal{R} of regular morphisms in $\mathcal{C}/\mathcal{X}_T$ admits a calculus of left fractions and a calculus of right fractions. Furthermore, the localisation $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ of $\mathcal{C}/\mathcal{X}_T$ at the class \mathcal{R} is abelian.*

Remark 5.2. *We have seen that the localisation $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ inherits an additive structure from $\mathcal{C}/\mathcal{X}_T$ (see Remark 4.3). The localisation $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ also inherits a k -additive structure from $\mathcal{C}/\mathcal{X}_T$: a scalar λ takes the fraction $[r, f]_{RF}$ to $[r, \lambda f]_{RF}$. It can be checked that this action is well defined and, together with the additive structure inherited from $\mathcal{C}/\mathcal{X}_T$, gives a k -additive structure on $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ for which the localisation functor is k -additive.*

We will show that the projectives in $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ are the objects in $\text{add } T$ and that $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ has enough projectives. From this it will follow that $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ is equivalent to $\text{mod } \Gamma$.

Lemma 5.3. *Let \mathcal{A} be an additive category and P a projective object in \mathcal{A} . If $r: U \rightarrow P$ is a regular morphism then it is an isomorphism.*

Proof. Since r is an epimorphism, the identity map on P factors through r , so there is a morphism $s: P \rightarrow U$ such that $rs = \text{id}$. Then $r(sr - \text{id}) = (rs)r - r = 0$. Since r is a monomorphism, $sr - \text{id} = 0$ and it follows that r is an isomorphism. \square

Lemma 5.4. *Let \mathcal{A} be a skeletally small integral category and \mathcal{R} the class of regular morphisms in \mathcal{A} . Suppose that P is a projective object in \mathcal{A} . Then P , when regarded as an object in the localisation $\mathcal{A}_{\mathcal{R}}$, is again a projective object.*

Proof. Suppose P is projective in \mathcal{A} and we have a diagram:

$$\begin{array}{ccc} & P & \\ & \downarrow [p]x_s & \\ X & \xrightarrow{[f]x_r} & Y \end{array}$$

in $\mathcal{A}_{\mathcal{R}}$ with p, f, r, s morphisms in \mathcal{A} , $s: U \rightarrow P$ and r regular morphisms and $[f]x_r$ an epimorphism in $\mathcal{A}_{\mathcal{R}}$. Since s is regular and P is projective in \mathcal{A} , s is an isomorphism by Lemma 5.3. Hence $P \simeq U$ in \mathcal{A} , so U is also projective in \mathcal{A} . By Lemma 4.5, f is an epimorphism in \mathcal{A} , so p factors through f . It follows that $[p]x_s$ factors through $[f]x_r$ as required. \square

Lemma 5.5. (a) *The projectives in $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ are exactly the objects in $\text{add } T$.*
 (b) *The category $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ has enough projectives.*

Proof. By Lemmas 5.4 and 3.5, the objects in $\text{add } T$ are projective in $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$. Let X be an object in $\mathcal{C}/\mathcal{X}_T$. Then, by Lemma 3.6, there is an epimorphism $\underline{p}: T_0 \rightarrow X$ in $\mathcal{C}/\mathcal{X}_T$, where T_0 lies in $\text{add } T$, and (b) follows, using Lemma 4.5. If X is projective, the identity map on X factors through $[\underline{p}]$, so $[\underline{p}]$ is a split epimorphism and X is isomorphic to a summand of T_0 , hence in $\text{add } T$, and (a) follows. \square

Lemma 5.6. *We have that $\text{End}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T) \simeq \text{End}_{\mathcal{C}}(T)$.*

Proof. The localisation functor induces a morphism

$$\varphi: \text{End}_{\mathcal{C}/\mathcal{X}_T}(T) \rightarrow \text{End}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T).$$

If $[\underline{f}]x_{\underline{r}}$ is an arbitrary element of $\text{End}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T)$ with $\underline{f}, \underline{r}$ morphisms in $\mathcal{C}/\mathcal{X}_T$ and \underline{r} regular, then \underline{r} is an isomorphism in $\mathcal{C}/\mathcal{X}_T$ by Lemma 5.3, so $x_{\underline{r}} = [\underline{r}^{-1}]$ and we see that φ is surjective. By Lemma 4.4, it is also injective, so

$$\text{End}_{\mathcal{C}/\mathcal{X}_T}(T) \simeq \text{End}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T).$$

The result follows, since the only homomorphisms from T to objects in \mathcal{X}_T are zero by definition. \square

Theorem 5.7. *Let \mathcal{C} be a skeletally small, Hom-finite, Krull-Schmidt triangulated category with Serre duality, containing a rigid object T . Let \mathcal{X}_T denote the class of objects X in \mathcal{C} such that $\text{Hom}_{\mathcal{C}}(T, X) = 0$. Let \mathcal{R} denote the class of regular morphisms in $\mathcal{C}/\mathcal{X}_T$ and $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ the localisation of the integral category $\mathcal{C}/\mathcal{X}_T$ at \mathcal{R} . Then*

$$(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}} \simeq \text{mod End}_{\mathcal{C}}(T)^{op}.$$

Proof. By Theorem 5.1, $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ is an abelian category. By Lemma 5.5, $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ has enough projectives, given by the objects in $\text{add } T$. The result follows, with an equivalence being given by the functor $\text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, -)$, noting that T is a projective generator for $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$. \square

We note that, by Remark 5.2, $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ inherits a k -additive structure from $\mathcal{C}/\mathcal{X}_T$. It is easy to see that the above equivalence preserves this structure (as well as the abelian structure).

6. COTORSION PAIRS

We recall the notion of a *cotorsion pair* in a triangulated category, considered in Nakaoka [14]. By [14, 2.3] this can be defined as a pair $(\mathcal{U}, \mathcal{V})$ of full additive subcategories satisfying

- (a) $\mathcal{U}^{\perp} = \mathcal{V}$;
- (b) $\mathcal{V}^{\perp} = \mathcal{U}$;
- (c) For any object C , there is a (not necessarily unique) triangle:

$$U \rightarrow C \rightarrow \Sigma V \rightarrow \Sigma U,$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Nakaoka points out that $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in this sense if and only if $(\mathcal{U}, \Sigma \mathcal{V})$ is a torsion theory in the sense of [8, 2.2].

If \mathcal{C} is a triangulated category as in Section 1 and T is a rigid object in \mathcal{C} then $(\text{add } T, T^{\perp})$ is a cotorsion pair (e.g. see [2, Sect. 6]). One might then ask whether Theorem 3.1 can be generalised to this set-up. However, it is easy to see that this cannot be the case. Consider a triangulated category \mathcal{C} , satisfying our usual assumptions. Assume that \mathcal{C} has a non-zero nonisomorphism between two indecomposables. Then this map does not have a cokernel or kernel in \mathcal{C} . But the pair $(\mathcal{U}, \mathcal{V}) = (\mathcal{C}, 0)$ is clearly a cotorsion pair. This gives many examples of cotorsion pairs $(\mathcal{U}, \mathcal{V})$ such that \mathcal{C}/\mathcal{V} is not preabelian.

More interesting examples where \mathcal{C}/\mathcal{V} is not preabelian also exist. Let \mathcal{C} be the cluster category associated to the quiver:

$$1 \leftarrow 2 \leftarrow 3.$$

For a vertex i , let P_i (respectively, I_i , S_i) denote the corresponding indecomposable projective (respectively, injective, simple) module. Let \mathcal{U} be the additive subcategory whose indecomposable objects are P_2, P_3 and ΣP_3 . It is easy to check that \mathcal{U}^{\perp} is the additive subcategory with indecomposables given by P_1, P_2 and S_2 and that $(\mathcal{U}, \mathcal{U}^{\perp})$ is a cotorsion pair. Note that the torsion pair $(\mathcal{U}, \Sigma \mathcal{U}^{\perp})$ appears in [7].

Let f be a non-zero map from P_3 to I_2 . Suppose that $c: I_2 \rightarrow C$ is a cokernel of f in $\mathcal{C}/\mathcal{U}^\perp$. Since the only non-zero maps $g: I_2 \rightarrow Y$ with Y indecomposable such that $gf = 0$ have $Y = \Sigma P_1$ or $Y = \Sigma P_2$, it follows that C is a direct sum of copies of ΣP_1 and ΣP_2 . Then $dc = 0$ for any non-zero map d from C to ΣP_3 , a contradiction to the fact that c is an epimorphism.

7. THE FUNCTOR $\text{Hom}_{\mathcal{C}}(T, -): \mathcal{C} \rightarrow \text{mod } \Gamma$

Let T denote a rigid object in a triangulated category \mathcal{C} , where \mathcal{C} satisfies the same properties as earlier, and let $\Gamma = \text{End}_{\mathcal{C}}(T)^{op}$. We have seen that we can obtain $\text{mod } \Gamma$ in a process consisting of two steps: first forming the preabelian factor category $\mathcal{C}/\mathcal{X}_T$, and then localising this category with respect to the class of regular morphisms.

In [2] we considered the functor $H = \text{Hom}_{\mathcal{C}}(T, -): \mathcal{C} \rightarrow \text{mod } \Gamma$. Let $\mathcal{S} = \mathcal{S}_T$ be the collection of maps $f: X \rightarrow Y$ in \mathcal{C} with the property that in the induced triangle

$$\Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z,$$

both g and h factor through \mathcal{X}_T . Let $L_{\mathcal{S}}: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{S}}$ denote the Gabriel-Zisman localisation. We proved in [2] that there is an equivalence $G: \mathcal{C}_{\mathcal{S}} \rightarrow \text{mod } \Gamma$ such that $GL_{\mathcal{S}} = H$. We also proved that a map s belongs to \mathcal{S} if and only if $H(s)$ is an isomorphism in $\text{mod } \Gamma$.

In this section, we point out that the functor H is actually naturally equivalent to the composition of the quotient functor $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{X}_T$ and the localisation functor with respect to regular morphisms.

Consider the set of maps $\mathcal{S}_0 = \{X \amalg U \rightarrow X \mid U \in \mathcal{X}_T, X \in \mathcal{C}\}$ and the Gabriel-Zisman localisation $L_{\mathcal{S}_0}: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{S}_0}$. The quotient functor $Q: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{X}_T$ inverts all maps in \mathcal{S}_0 , so there is a functor $G_0: \mathcal{C}_{\mathcal{S}_0} \rightarrow \mathcal{C}/\mathcal{X}_T$, making the following diagram commute.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{Q} & \mathcal{C}/\mathcal{X}_T \\ & \searrow L_{\mathcal{S}_0} & \nearrow G_0 \\ & \mathcal{C}_{\mathcal{S}_0} & \end{array}$$

The localisation functor $L_{\mathcal{S}_0}$ has the following elementary properties.

- Lemma 7.1.** (a) For U in \mathcal{X}_T , consider the projection map $\pi_X: X \amalg U \rightarrow X$. The inverse of $L_{\mathcal{S}_0}(\pi_X)$ is $L_{\mathcal{S}_0}(\iota_X)$, where $\iota_X: X \rightarrow X \amalg U$ is the canonical inclusion map.
- (b) Let u, v be maps in \mathcal{C} such that v factors through \mathcal{X}_T . Then $L_{\mathcal{S}_0}(u + v) = L_{\mathcal{S}_0}(u)$ in $\mathcal{C}_{\mathcal{S}_0}$.

Proof. The proof is identical to the proof of Lemma 3.5 in [2]. \square

By construction, G_0 is the identity on objects. It is clear that G_0 is full, since Q has this property.

We claim that G_0 is also faithful. Firstly, note that by Lemma 7.1(a), $L_{\mathcal{S}_0}$ is full. So let f and f' be maps in \mathcal{C} with $G_0 L_{\mathcal{S}_0}(f) = G_0 L_{\mathcal{S}_0}(f')$. Then $H(f) = H(f')$, so $f - f'$ factors through \mathcal{X}_T (by [2, Lemma 2.3]), and hence, by Lemma 7.1, we have that $L_{\mathcal{S}_0}(f) = L_{\mathcal{S}_0}(f' + f - f') = L_{\mathcal{S}_0}(f')$. Hence, we have the following.

Proposition 7.2. *The induced functor $G_0: \mathcal{C}_{\mathcal{S}_0} \rightarrow \mathcal{C}/\mathcal{X}_T$ is an isomorphism of categories.*

By Lemma 3.3, a morphism \underline{f} in $\mathcal{C}/\mathcal{X}_T$ is regular if and only if f lies in \mathcal{S} . Combining this with Proposition 7.2 we see that the image $L_{\mathcal{S}_0}(\mathcal{S})$ in the preabelian category $\mathcal{C}_{\mathcal{S}_0}$ consists of exactly the regular morphisms.

Let \mathcal{R} denote the regular morphisms in $\mathcal{C}/\mathcal{X}_T$. By the universal property of localisation, it follows that we also get an induced isomorphism of categories

$$K: (\mathcal{C}_{\mathcal{S}_0})_{L_{\mathcal{S}_0}(\mathcal{S})} \rightarrow (\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$$

making the following diagram commute.

$$\begin{array}{ccc} \mathcal{C}/\mathcal{X}_T & \xrightarrow{L_{\mathcal{R}}} & (\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}} \\ \uparrow G_0 & & \uparrow K \\ \mathcal{C}_{\mathcal{S}_0} & \xrightarrow{L_{L_{\mathcal{S}_0}(\mathcal{S})}} & (\mathcal{C}_{\mathcal{S}_0})_{L_{\mathcal{S}_0}(\mathcal{S})} \end{array} \quad \begin{array}{c} G_0^{-1} \\ K^{-1} \end{array}$$

It is clear that H factors uniquely through Q , and hence by the universal property of localisation, also uniquely through $L_{\mathcal{R}}: \mathcal{C}/\mathcal{X}_T \rightarrow (\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$, so we have a commutative diagram of functors

$$\begin{array}{ccccc} & & H & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{C} & \xrightarrow{Q} & \mathcal{C}/\mathcal{X}_T & \xrightarrow{L_{\mathcal{R}}} & (\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}} & \xrightarrow{H'} & \text{mod End}_{\mathcal{C}}(T)^{op} \end{array}$$

Lemma 7.3. *The functor H' is naturally isomorphic to the functor $\text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, -)$ which gives an equivalence between $(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}$ and $\text{mod End}_{\mathcal{C}}(T)^{op}$ in Theorem 5.7.*

Proof. Firstly, we note that the map $\varphi_X: f \mapsto \underline{[f]}$ gives an isomorphism from $H'(X) = \text{Hom}_{\mathcal{C}}(T, X)$ to $\text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, X)$ as $\text{End}_{\mathcal{C}}(T)^{op}$ -modules (arguing as in the proof of Lemma 5.6). Secondly, let $u = x_{\underline{r}}[\underline{f}] \in \text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(X, Y)$ be an arbitrary morphism, where $f: X \rightarrow Z$ and $r: Y \rightarrow Z$ are morphisms in \mathcal{C} for some object Z in \mathcal{C} and \underline{r} is regular in $\mathcal{C}/\mathcal{X}_T$. Consider the diagram:

$$\begin{array}{ccc} H'(X) & \xrightarrow{\varphi_X} & \text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, X) \\ \downarrow H'(u) & & \downarrow \text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, u) \\ H'(Y) & \xrightarrow{\varphi_Y} & \text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, Y). \end{array}$$

Let $\alpha \in H'(X) = \text{Hom}_{\mathcal{C}}(T, X)$. Then:

$$\begin{aligned}
 H'(u)(\alpha) &= H'(x_r[\underline{f}])(\alpha) \\
 &= H'(x_r)H'([\underline{f}])(\alpha) \\
 &= H'(x_r)H(f)(\alpha) \\
 &= H'(x_r)(f\alpha) \\
 &= \text{Hom}_{\mathcal{C}}(T, r)^{-1}(f\alpha) = g,
 \end{aligned}$$

where $f\alpha = rg$. Hence $\varphi_Y(H'(u)(\alpha)) = [g]$. We also have

$$\begin{aligned}
 \text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, u)(\varphi_X(\alpha)) &= \text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}}}(T, x_r[\underline{f}])([\underline{\alpha}]) \\
 &= x_r[\underline{f}][\underline{\alpha}] \\
 &= x_r[\underline{r}][\underline{g}] = [\underline{g}],
 \end{aligned}$$

so the diagram commutes and the lemma is proved. \square

It follows that H' is an equivalence.

We also have a commutative diagram of functors

$$\begin{array}{ccccc}
 \mathcal{C} & \xrightarrow{L_{S_0}} & \mathcal{C}_{S_0} & \xrightarrow{L_{L_{S_0}(S)}} & (\mathcal{C}_{S_0})_{L_{S_0}(S)} & \xrightarrow{L'} & \mathcal{C}_S \\
 & & & & \text{L}_S & &
 \end{array}$$

in which L' is an isomorphism of categories. This follows from the universal property satisfied by the localisation functors involved.

Let G^{-1} denote a quasi-inverse of G . Summarising, we have:

Proposition 7.4. *We have the following diagram of functors. The diagram commutes, apart from the rightmost square, which commutes only up to natural isomorphism.*

$$\begin{array}{ccccccc}
 & & & & H & & \\
 & & & & \curvearrowright & & \\
 \mathcal{C} & \xrightarrow{Q} & \mathcal{C}/\mathcal{X}_T & \xrightarrow{L_{\mathcal{R}}} & (\mathcal{C}/\mathcal{X}_T)_{\mathcal{R}} & \xrightarrow[\cong]{H'} & \text{mod End}_{\mathcal{C}}(T)^{op} \\
 & \searrow L_{S_0} & \uparrow G_0 & & \uparrow K & & \uparrow G \\
 & & \downarrow G_0^{-1} & & \downarrow K^{-1} & & \downarrow G^{-1} \\
 & & \mathcal{C}_{S_0} & \xrightarrow{L_{L_{S_0}(S)}} & (\mathcal{C}_{S_0})_{L_{S_0}(S)} & \xrightarrow[\cong]{L'} & \mathcal{C}_S \\
 & & & & \text{L}_S & &
 \end{array}$$

Proof. We have checked above that the diagram commutes apart from the rightmost square. We recall that, for a localisation functor L and two functors J, J' composable with it, $JL = J'L$ implies that $J = J'$, by the universal property. We have that

$$GL'L_{L_{S_0}(S)}L_{S_0} = GL_S = H = H'L_{\mathcal{R}}Q = H'L_{\mathcal{R}}G_0L_{S_0} = H'KL_{L_{S_0}(S)}L_{S_0},$$

so $GL' = H'K$. It follows that $G^{-1}H'$ is naturally equivalent to $L'K^{-1}$. \square

We remark that the fact that L' is an isomorphism implies that H' is an equivalence of categories, by the commutativity of the right hand square. Thus the equivalence in Theorem 5.7 can also be derived from the fact that G is an equivalence (i.e. [2, Theorem 4.3]) together with the above analysis.

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REFERENCES

- [1] Beligiannis, Apostolos; Reiten, Idun *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. 188 (2007), no. 883.
- [2] Buan, Aslak Bakke and Marsh, Robert J. *From triangulated categories to module categories via localisation*, Preprint arXiv:1010.0351v1 [math.RT], 2010. To appear in Trans. Amer. Math. Soc.
- [3] Buan, Aslak Bakke; Marsh, Robert J.; Reiten, Idun *Cluster-tilted algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 1, 323–332
- [4] Gabriel, P.; Zisman, M. *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35 Springer-Verlag New York, Inc., New York 1967.
- [5] Gelfand, Sergei I.; Manin, Yuri I. *Methods of homological algebra*, Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
- [6] Happel, D. *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.
- [7] Holm, Thorsten; Jørgensen, Peter; Rubey, Martin *Ptolemy diagrams and torsion pairs in the cluster category of Dynkin type A_n* , Preprint arXiv:1010.1184v2 [math.RT], 2010. To appear in J. Algebraic Combin.
- [8] Iyama, Osamu; Yoshino, Yuji *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. 172 (2008), no. 1, 117–168.
- [9] Keller, Bernhard *Derived categories and their uses*, Handbook of algebra, Vol. 1, 671701, North-Holland, Amsterdam, 1996.
- [10] Keller, Bernhard; Reiten, Idun *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. 211 (2007), no. 1, 123–151.

- [11] Koenig, Steffen; Zhu, Bin *From triangulated categories to abelian categories: cluster tilting in a general framework*, Math. Z. 258 (2008), no. 1, 143–160.
- [12] Krause, Henning *Localization theory for triangulated categories*, in Triangulated Categories (edited by Holm, T.; Jørgensen, P.; Rouquier, R.), London Math. Soc. Lecture Note Ser., vol. 375, Cambridge University Press, Cambridge, 2010.
- [13] Mac Lane, Saunders *Categories for the working mathematician*, Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998.
- [14] Nakaoka, Hiroyuki *General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories*, Appl. Categ. Structures, Online First 2010, DOI: 10.1007/s10485-010-9223-2.
- [15] Osborne, M. Scott *Basic homological algebra*, Graduate Texts in Mathematics, 196. Springer-Verlag, New York, 2000.
- [16] Rump, Wolfgang *Categories of lattices, and their global structure in terms of almost split sequences*, Algebra Discrete Math. 2004, no. 1, 87–111.
- [17] Rump, Wolfgang *Almost abelian categories*, Cahiers Topologie Géom. Différentielle Catég 42 (2001), no. 3, 163–225.
- [18] Wakamatsu, Takayoshi *On modules with trivial self-extensions*, J. Algebra 114 (1988), no. 1, 106–114.
- [19] Weibel, Charles A. *An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38*, Cambridge University Press, Cambridge, 1994.

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